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6. AUTHOR(S)
A.A. Shabana and R.Y. Yakoub

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Chicago, IL 60607-7022

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Department of Mechanical Engineering

University of Illinois at Chicago

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AN ISOPARAMETRIC THREE DIMENSIONAL BEAM ELEMENT USING THE ABSOLUTE NODAL COORDINATE FORMULATION

A. A. Shabana

R. Y. Yakoub

Department of Mechanical Engineering

University of Illinois at Chicago

842 West Taylor Street

Chicago, IL 60607-7022

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ABSTRACT

The description of a beam element by only the displacement of its center line leads to some difficulties in the representation of the torsion and shear effects. For instance such a representation does not capture the rotation of the beam as a rigid body about its own axis. This problem was circumvented in the literature by using a local coordinate system in the incremental finite element method or by using the multibody floating frame of reference formulation. The use of such a local element coordinate system leads to a highly nonlinear expression for the inertia forces as the result of the large element rotation. In this investigation, an absolute nodal coordinate formulation is presented for the large rotation and deformation analysis of three dimensional beam elements. This formulation leads to a constant mass matrix, and as a result, the vectors of the centrifugal and Coriolis forces are identically equal to zero. The formulation presented in this paper takes into account the effect of rotary inertia, torsion and shear effects, and ensures continuity of the slopes as well as the rotation of the beam cross section at the nodal points. Using the proposed formulation curved beams can be systematically modeled.

1 INTRODUCTION

In the two dimensional analysis, the center line of a beam element can be used to completely describe the beam kinematics according to Euler-Bernoulli beam assumptions. A vector that defines the location of an arbitrary point on the beam cross section in terms of the spatial longitudinal coordinate is sufficient for the determination of the orientation and the position vector of the origin of a Frenet frame [5, 12], which has one of its axes tangent to the center line, and the other two axes are perpendicular to the center line. The effect of the rotary inertia of the beam cross section can be systematically accounted for using Frenet frame whose orientation is completely defined by the derivatives of the displacement vector. This was the basis for developing the absolute nodal coordinate formulation [11, 16, 18, 20], which does not require the interpolation of finite rotation coordinates as it is the case in some finite element procedures. It was also shown that the absolute nodal coordinate formulation can account for the rotary inertia effect and at the same time the mass matrix remains constant [16, 23]. The fact that the mass matrix remains constant becomes crucial in developing an efficient algorithm for solving the multibody dynamic equations. Using this property, an optimum sparse matrix structure can be obtained using Cholesky coordinates [19, 24].

The problem of three dimensional beams requires more careful consideration since the motion of the beam, even within the Euler-Bernoulli beam assumptions, can not be completely described by the displacement of its center line. The center line represents a spatial curve, as shown in Fig. 1. The location of an arbitrary point on this spatial curve in a coordinate system XYZ is defined by the vector $\mathbf{r}(s)$, where s is the arc length. While the vector $\mathbf{r}(s)$ can be used to define a Frenet frame which has three orthogonal vectors; tangent, normal, and binormal, such a representation fails to capture simple beam motion. For instance, the rotation of the beam about its cross section as a rigid body can not be described by the vector $\mathbf{r}$ since this vector remains constant throughout this simple motion. This problem has been avoided in the finite element and multibody literature by introducing

a local coordinate system. In the finite element literature, convected coordinate systems are used for the finite element [1, 15]. The deformation of the element can be defined in the convected system whose orientation can be described using three independent parameters. In the multibody literature, the floating frame of reference formulation is used [2, 6, 9, 17]. In this formulation, a coordinate system is introduced for the flexible body. This coordinate system can be used to describe the gross body motion. The body deformation is defined in the body coordinate system which can capture the rotation of the beam element about its own axis. Another benefit of using the floating frame of reference formulation is the exact representation of the rigid body motion even when conventional non-isoparametric elements such as beams and plates are used. With this formulation, the finite element has zero strain under an arbitrary rigid body motion.

It is important to point out that most of existing finite element formulations, including large rotation vector formulations [3, 21, 22], lead to a highly nonlinear mass matrix when large rotation and deformation of three dimensional beam elements are considered. It is the objective of this investigation to develop an absolute nodal coordinate formulation for three dimensional beams that undergo large rotations and deformations. It is shown that the mass matrix of the element is constant, and as a consequence, the centrifugal and Coriolis inertia forces are identically equal to zero. The effects of the rotary inertia, torsion and shear are automatically accounted for. The formulation of the mass matrix and the elastic forces is presented and the choice of the element nodal coordinates is discussed.

2 DISPLACEMENT FIELD

The order of the polynomial used in the assumed displacement fields in the finite element analysis depends on the expected shape of the element deformation. In the conventional finite element analysis, a linear displacement field is assumed to interpolate the longitudinal

deformation, while a cubic polynomial is used for the transverse deflections of the beam element. This kind of assumed displacement field is suitable when using a beam local coordinate system as in the case of the incremental procedure or the finite element floating frame of reference formulation that is widely used in flexible multibody simulation. However, in the non-incremental absolute nodal coordinate formulation, the concept is different since the chosen displacement field represents both rigid and flexible body motions. This displacement field is defined in the global system and accounts for the coupling between the rigid body motion and the elastic deformation. For a three dimensional beam element, we assume a displacement field using the following polynomials:

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} a_0 + a_1 x + a_2 y + a_3 z + a_4 xy + a_5 xz + \sum\limits_{k=6}^{n} a_k x^{k-4} \\ b_0 + b_1 x + b_2 y + b_3 z + b_4 xy + b_5 xz + \sum\limits_{k=6}^{n} b_k x^{k-4} \\ c_0 + c_1 x + c_2 y + c_3 z + c_4 xy + c_5 xz + \sum\limits_{k=6}^{n} c_k x^{k-4} \end{bmatrix}, \qquad (1)$$

where $\mathbf{r}$ is the global position vector of an arbitrary point on the beam cross section, a_i, b_i, and c_i are the polynomial coefficients, and x, y and z are the spatial coordinates defined in a chosen beam coordinate system. Here x is assumed to be the spatial coordinate along the beam axis $(0 \leqslant x \leqslant l)$, where l is the length of the beam element. The order of the polynomials in the preceding equation can be chosen depending on the magnitude of the deformation expected from the element. Note that the polynomials used to interpolate the three components of the displacements have the same order since the vector $\mathbf{r}$ is defined in the global coordinate system. Furthermore, in order to account for the rotary inertia and shear effects, the displacement field is assumed to depend on y and z. Since the cross section dimensions of the beam element are assumed to be small compared to the element length, the displacement field is assumed to depend only linearly on the spatial coordinates y and

z. The preceding equation can also be written in a matrix form as follows:

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S}_1 \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix}, \tag{2}$$

where $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are the vectors of the polynomial coefficients, and $\mathbf{S}_1$ is a row vector that defines the space-dependent coefficients of the polynomials of Eq. (1).

The order of the interpolating polynomials used for the assumed displacement field depends on the chosen number of the beam nodal coordinates. The choice of the nodal coordinates is as important as the choice of the displacement fields since it contributes to the definition of the element stiffness and inertia forces. A combination of global position and slope coordinates can always be chosen as nodal coordinates as will be demonstrated in details in later sections of this paper.

3 DEFINITION OF BEAM CROSS SECTION

In Euler-Bernoulli beam theory, the cross section is assumed to remain rigid and perpendicular to the center line. Therefore, the normal to the cross section is defined by the tangent vector $\partial\mathbf{r}/\partial x$. In the assumed displacement field defined in the preceding section, the beam cross section does not remain perpendicular to the center line, and therefore, the cross section is not defined by the vector $\partial\mathbf{r}/\partial x$ tangent to the center line. Using this displacement field, it can be shown that for a given x, $\dfrac{\partial\mathbf{r}}{\partial y}$ and $\dfrac{\partial\mathbf{r}}{\partial z}$ are independent of y and z. It can also be shown that $\dfrac{\partial\mathbf{r}}{\partial y}$ and $\dfrac{\partial\mathbf{r}}{\partial z}$ are two independent vectors (not necessarily orthogonal) that define the cross section of the beam element. To this end, an arbitrary vector $\mathbf{r}_s$ is defined in the cross section as shown in Fig. 2. Using the displacement field defined in the preceding

section, it can be shown that

$$\mathbf{r}_s = \mathbf{r}_P - \mathbf{r}_{\bar{P}} = \begin{bmatrix} a_2 + xa_4 \\ b_2 + xb_4 \\ c_2 + xc_4 \end{bmatrix} y + \begin{bmatrix} a_3 + xa_5 \\ b_3 + xb_5 \\ c_3 + xc_5 \end{bmatrix} z, \tag{3}$$

where $\mathbf{r}_P$ is the global position vector of an arbitrary point P on the cross section with coordinates (x, y, z), and $\mathbf{r}_{\bar{P}}$ is the global position vector of a corresponding point $\bar{P}$ with coordinates $(x, 0, 0)$ on the center line of the beam.

It can be shown that

$$\frac{\partial \mathbf{r}}{\partial y} = \begin{bmatrix} a_2 + xa_4 \\ b_2 + xb_4 \\ c_2 + xc_4 \end{bmatrix}, \qquad \frac{\partial \mathbf{r}}{\partial z} = \begin{bmatrix} a_3 + xa_5 \\ b_3 + xb_5 \\ c_3 + xc_5 \end{bmatrix}. \tag{4}$$

The preceding two equations (Eqs. 3 and 4) show that

$$\mathbf{r}_s = y\frac{\partial \mathbf{r}}{\partial y} + z\frac{\partial \mathbf{r}}{\partial z},$$

which shows that an arbitrary vector drawn on the cross section can be expressed as a linear combination of the two vectors $\frac{\partial \mathbf{r}}{\partial y}$ and $\frac{\partial \mathbf{r}}{\partial z}$. The fact that $\frac{\partial \mathbf{r}}{\partial y}$ and $\frac{\partial \mathbf{r}}{\partial z}$ are independent of y and z can be used, with the help of Gram-Shmidt orthogonalization process [12], to determine the following two orthonormal vectors on the cross section:

$$\mathbf{n}_s = \frac{\mathbf{r}_y}{|\mathbf{r}_y|}, \qquad \mathbf{b}_s = \frac{\mathbf{r}_z - h\mathbf{r}_y}{|\mathbf{r}_z - h\mathbf{r}_y|}, \tag{5}$$

where

$$h = \frac{(\mathbf{r}_z)^T \mathbf{r}_y}{(\mathbf{r}_y)^T \mathbf{r}_y}. \tag{6}$$

$\mathbf{n}_s$ is a unit vector along $\mathbf{r}_y = \frac{\partial \mathbf{r}}{\partial y}$, while $\mathbf{b}_s$ is a unit vector which is a linear combination of the vectors $\mathbf{r}_z = \frac{\partial \mathbf{r}}{\partial z}$ and $\mathbf{n}_s$. In order to complete the orthogonal triad for the cross section, a normal to the beam cross section can be obtained by using the cross product of the vectors $\mathbf{n}_s$ and $\mathbf{b}_s$ as follows:

$$\mathbf{t}_s = \mathbf{n}_s \times \mathbf{b}_s \tag{7}$$

The vectors $\mathbf{t}_s, \mathbf{n}_s,$ and $\mathbf{b}_s$ constitute a right-handed orthonormal set, which completely defines the orientation of the beam cross section.

4 KINEMATICS OF THE CROSS SECTION

As described in the preceding section, the three orthogonal vectors $\mathbf{t}_s, \mathbf{n}_s$, and $\mathbf{b}_s$ can be systematically defined in terms of the vectors $\mathbf{r}_y$ and $\mathbf{r}_z$ obtained using the displacement field introduced in Section 2. In the case of a rigid body motion of the finite element it can be shown that $\mathbf{r}_y$ and $\mathbf{r}_z$ are two orthogonal unit vectors, and as a consequence, h in Eq. (6) is identically equal to zero, and

$$\mathbf{n}_s = \mathbf{r}_y, \text{ and } \qquad \mathbf{b}_s = \mathbf{r}_z. \tag{8}$$

Furthermore, in the case of rigid body motion of a straight beam, $\mathbf{t}_s$ remains in the direction of the center line of the beam. Therefore, in the case of a general displacement that includes deformations, the difference between $\mathbf{n}_s$ and $\mathbf{r}_y$, and $\mathbf{b}_s$ and $\mathbf{r}_z$ is mainly due to the beam deformation. In fact h in Eq. (6) can be used as a measure of this difference since h is identically equal to zero under an arbitrary rigid body motion. If the deformation within the element is small, then it is reasonable to assume that

$$h \approx 0,$$

and in this case

$$\mathbf{n}_s \approx \mathbf{r}_y, \text{ and } \qquad \mathbf{b}_s \approx \mathbf{r}_z.$$

Using this assumption, it follows from the definition of $\mathbf{r}_s$ presented in the preceding section that

$$|\mathbf{r}_s| \approx constant,$$

which implies that the cross section remains rigid.

It is important to point out that the assumption used in this section is also commonly used in the finite element literature for beam elements that account for the effect of rotary inertia and torsion [14] as will be discussed in Section 9.

5 MEASURES OF TORSION AND SHEARS

In Euler-Bernoulli beam theory, it is assumed that the cross section of the beam remains perpendicular to the beam center line, and as a result, the shear deformation effect is neglected. In this case, the vector normal to the cross section coincides with the tangent to the center line defined by Serret-Frenet equations.

Serret-Frenet frame is a coordinate system which defines three vectors; tangent, normal, and binormal to the beam center line. The normal and binormal vectors define the so called normal plane [12] which is normal to the center line. The normal plane of Serret-Frenet frame is, therefore, the beam cross section if Euler-Bernoulli beam theory is used. The Serret-Frenet frame is defined by three orthogonal unit vectors; $\mathbf{t}, \mathbf{n}$, and $\mathbf{b}$. The unit vector $\mathbf{t}$ is tangent to the beam center line and is defined as

$$\mathbf{t} = \frac{d\mathbf{r}}{ds}, \tag{9}$$

where ds is the arc length of an infinitesimal segment of the center line defined as

$$ds = \sqrt{(\mathbf{r}_x)^T \mathbf{r}_x} dx, \tag{10}$$

where $\mathbf{r}_x = \dfrac{\partial \mathbf{r}}{\partial x}$. Thus, Eq. (9) can be written as

$$\mathbf{t} = \frac{\mathbf{r}_x}{|\mathbf{r}_x|} = \frac{\mathbf{r}_x}{\sqrt{(\mathbf{r}_x)^T \mathbf{r}_x}}. \tag{11}$$

The unit vector $\mathbf{n}$ is normal to the beam center line and is defined as

$$\mathbf{n} = \frac{1}{\kappa}\frac{d\mathbf{t}}{ds} = \frac{1}{\kappa}\frac{d\mathbf{t}}{dx}\frac{dx}{ds}, \tag{12}$$

where κ defines the curvature of the deformed centerline. Equation (12) leads to

$$\mathbf{n} = \frac{1}{\kappa} \frac{\mathbf{r}_{xx} - h_f \mathbf{r}_x}{(\mathbf{r}_x)^T \mathbf{r}_x}, \tag{13}$$

where $\mathbf{r}_{xx} = \dfrac{\partial^2 \mathbf{r}}{\partial x^2}$, and h_f is a scalar quantity defined as

$$h_f = \frac{(\mathbf{r}_{xx})^T \mathbf{r}_x}{(\mathbf{r}_x)^T \mathbf{r}_x}. \tag{14}$$

The curvature can be written as

$$\kappa = \left| \frac{\mathbf{r}_{xx} - h_f \mathbf{r}_x}{(\mathbf{r}_x)^T \mathbf{r}_x} \right| = \left| \frac{d\mathbf{t}}{ds} \right| = \left| \frac{d^2 \mathbf{r}}{ds^2} \right|. \tag{15}$$

The unit vectors $\mathbf{t}$ and $\mathbf{n}$ define a plane, called the osculating plane, which represents the bending plane of the center line of the beam. However, the orientation of this plane is a function of the beam axial coordinate x. The third vector that completes the Serret-Frenet orthogonal triad is the binormal unit vector $\mathbf{b}$ defined as

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}. \tag{16}$$

The rate at which the curve twists out of its osculating plane is called the *torsion* τ, which is defined as [12]

$$\tau = \left| \frac{d\mathbf{b}}{ds} \right|, \tag{17}$$

and it represents the twisting shear deformation of the beam. However, as pointed out in the introduction of this paper, the Serret-Frenet description fails to capture the rotation of the cross section of a straight beam about its own axis since such a rotation does not contribute to the change of the vector $\mathbf{r}$.

As already mentioned, the plane defined by $\{\mathbf{n}, \mathbf{b}\}$ is perpendicular to the center line of the beam, while the plane defined by $\{\mathbf{n}_s, \mathbf{b}_s\}$ represents the cross section of the beam defined by the assumed displacement field presented in Section 2. In the undeformed beam configuration (rigid body motion), or in the case of no shear effect, the two planes are parallel.

In the case of a general displacement, the two planes are not parallel due shear deformations. The scalar

$$v = \mathbf{t} \cdot \mathbf{t}_s = \mathbf{t} \cdot (\mathbf{n}_s \times \mathbf{b}_s) = \det \begin{bmatrix} \mathbf{t}^T \\ \mathbf{n}_s^T \\ \mathbf{b}_s^T \end{bmatrix}, \tag{18}$$

gives an idea about the shear deformation of the beam as shown in Fig. 3. For instance, if $v = 1$, there is no shear deformation, and if it is not equal to one, the amount of shear is inversely proportional to v $(-1 \leqslant v \leqslant 1)$. Figure 4 illustrates a general deformation of the beam by assuming arbitrary values for the polynomial coefficients. Also shown, in Fig. 4, is the angle γ which is the arc cosine of v at two different points on the beam $(\xi = x/l)$. Later in this paper, the strain energy due to the shear effect will be discussed.

Using the assumed displacement field of the beam presented in Section 2, the effect of the torsion can be easily accounted for using the two unit vectors $\mathbf{n}_s$ and $\mathbf{b}_s$ of the beam cross section. As a measure of torsion for a short beam, one can use the rotation of the cross section at an arbitrary point on the beam center line with respect to the reference plane at node A. There exists an angle β_x defined by the dot product

$$\cos \beta_x = \mathbf{n}_s \cdot \mathbf{n}_{sA}, \qquad \sin \beta_x = \mathbf{n}_s \cdot \mathbf{b}_{sA}, \tag{19}$$

where $\mathbf{n}_{sA}$ and $\mathbf{b}_{sA}$ are the two unit vectors that define the reference plane. Note that β_x is equal to zero in the case of rigid body motion, and as a consequence, it can be used as a measure for the torsion.

6 BEAM INERTIA

In the three dimensional analysis of the large rotation problem, both the finite element incremental approach and large rotation vector formulation lead to a highly nonlinear mass matrix. The absolute nodal coordinate formulation on the other hand leads to a constant

mass matrix and automatically accounts for the shear and torsional effects.

The mass matrix of the three dimensional beam element can be obtained using the absolute nodal coordinate formulation and the following expression of the kinetic energy:

$$T = \frac{1}{2} \int\limits_V \rho \dot{\mathbf{r}}^T \dot{\mathbf{r}} dV, \tag{20}$$

where $\mathbf{r}$ is the global position vector of an arbitrary point, and ρ and V are respectively the mass density and volume of the beam element. The vector $\mathbf{r}$ is defined in Section 2 by Eq. (1) using the coefficients of the interpolation polynomials. These coefficients can be replaced by global coordinates and slopes at the nodes, as explained in later sections of this paper. In this case, the global position vector of the arbitrary point can be expressed in terms of a vector of element nodal coordinates $\mathbf{e}$ and the shape function $\mathbf{S}$ as

$$\mathbf{r} = \mathbf{S}\mathbf{e}, \tag{21}$$

It follows that

$$\dot{\mathbf{r}} = \mathbf{S}\dot{\mathbf{e}}. \tag{22}$$

By substituting Eq. (22) into Eq. (20), one obtains

$$T = \frac{1}{2} \dot{\mathbf{e}}^T \left[\int\limits_V \rho \mathbf{S}^T \mathbf{S} dV \right] \dot{\mathbf{e}}, \tag{23}$$

which is a simple quadratic form in the velocities. Thus the element mass matrix is defined as

$$\mathbf{M} = \int\limits_V \rho \mathbf{S}^T \mathbf{S} dV. \tag{24}$$

The above integration defines a constant mass matrix which only depends on the inertia properties and dimensions of the beam. Using the fact that the mass matrix obtained using the absolute nodal coordinate formulation is constant, efficient numerical procedures can be used to obtain an optimum sparse matrix structure for the resulting multibody dynamic equations [19, 24]

The inclusion of the effect of the rotary inertia of the three dimensional formulation presented in this paper can be demonstrated by using Eq. (3). To this end, we write

$$\dot{\mathbf{r}}_P = \dot{\mathbf{r}}_{\bar{P}} + \dot{\mathbf{r}}_s, \tag{25}$$

where $\bar{P}$ is an arbitrary point on the beam center line. Using the preceding equation, the beam kinetic energy can be written as

$$\begin{aligned} T &= \frac{1}{2}\int_V \rho \left(\dot{\mathbf{r}}_{\bar{P}}^T + \dot{\mathbf{r}}_s^T\right)\left(\dot{\mathbf{r}}_{\bar{P}} + \dot{\mathbf{r}}_s\right) dV \qquad (26) \\ &= \frac{1}{2}\left\{ \int_l \frac{m}{l}\dot{\mathbf{r}}_{\bar{P}}^T\dot{\mathbf{r}}_{\bar{P}} dx + \int_V \rho \left[2\dot{\mathbf{r}}_{\bar{P}}^T\dot{\mathbf{r}}_s + \dot{\mathbf{r}}_s^T\dot{\mathbf{r}}_s\right] dxdydz \right\}. \end{aligned}$$

The first term in the above integration is the mass matrix in the case in which the beam rotary inertia is neglected, while the second term accounts for the effect of the rotary inertia.

7 ELASTIC FORCES

If the deformation within the beam is large, the expression for the nonlinear strain-displacement relationship must be used to formulate the elastic forces of the beam element [4] in the non-incremental absolute nodal coordinate formulation. However, if the size of the element is chosen to be small such that the deformation within the element remains small, the linear strain-displacement relationship can be used in the large deformation analysis of flexible bodies using the absolute nodal coordinate formulation [20, 24]. In this section, as an example for formulating the elastic forces, the case of small element deformation is considered.

In the finite element analysis, the strain energy of a beam is defined, in general, in terms of six components; one axial force, two bending moments, two shear forces, and one torsional moment [14]. However, in Euler-Bernoulli beam theory, the shear forces are not considered assuming that the cross section plane remains perpendicular to the beam centerline after

deformation. Thus, the strain energy in the case of small deformation can be defined as [10]

$$U = \frac{1}{2}\int_0^l \left\{ \begin{array}{c} EA\left(\frac{\partial u_x}{\partial x}\right)^2 + EI_{yy}\left(\frac{\partial^2 u_y}{\partial y^2}\right)^2 + EI_{zz}\left(\frac{\partial^2 u_z}{\partial z^2}\right)^2 + \\ Gk\beta_y^2 + Gk\beta_z^2 + GI_{xx}\left(\frac{\partial \beta_x}{\partial x}\right)^2 \end{array} \right\} dx, \tag{27}$$

where u_x, u_y, and u_z are respectively the $x-$, $y-$, and $z-$component of the beam deflection, β_x, β_y, and β_z are the shear angles, k is the Timoshenko shear factor [8, 10], E and G are respectively the moduli of elasticity and rigidity, I_{xx}, I_{yy}, and I_{zz} are the second moments of area, and A is the beam cross section area. In order to evaluate the beam strain energy, a local beam frame can be used. However, it is important to point out that the beam strain energy can be also evaluated using the inertial coordinate system directly instead of using a local frame by utilizing continuum mechanics theories [4, 21, 22, 23].

Previously, two coordinate systems were introduced in this paper; the cross section frame and the Serret-Frenet frame. In order to define the beam longitudinal and transverse deflections, a vector $\mathbf{d}$, shown in Fig. 5, is defined as

$$\mathbf{d} = \mathbf{r}_{\bar{P}} - \mathbf{r}_A, \tag{28}$$

where $\mathbf{r}_{\bar{P}}$ is the position vector of an arbitrary point $\bar{P}$ on the beam centerline, and $\mathbf{r}_A$ is the position vector of node A as shown in Fig. 5. Considering Eq. (21), Eq. (28) can be re-written as

$$\mathbf{d} = (\mathbf{S}(x,0,0) - \mathbf{S}(0,0,0))\,\mathbf{e}. \tag{29}$$

As shown in Fig. 5, the vector $\mathbf{d}$ represents the location of a point $\bar{P}$ on the beam center line with respect to node A, the deflection components can be defined as follows:

$$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \mathbf{A}^T\mathbf{d} - \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}, \tag{30}$$

where x is the axial coordinate of point $\bar{P}$ in the undeformed state, $\mathbf{A}$ is a transformation matrix which can be expressed in terms of the unit vectors $\mathbf{t}_s, \mathbf{n}_s$, and $\mathbf{b}_s$ defined at node A.

It follows from the preceding equation that

$$\left.\begin{aligned} u_x &= \mathbf{t}_{sA} \cdot \mathbf{d} - x \\ u_y &= \mathbf{n}_{sA} \cdot \mathbf{d} \\ u_z &= \mathbf{b}_{sA} \cdot \mathbf{d} \end{aligned}\right\} . \tag{31}$$

Equation (31) defines the axial and transverse components of the beam deflection which are required for evaluating the first three terms of the strain energy.

There are different approaches that can be used to measure the torsion and shear. Figures 6a, b, and c show a schematic diagram for each of the three rotation angles of the beam cross section. Using the difference in orientation between the cross section frame and Serret-Frenet frame depicted in these figures, one can conclude the following:

$$\left.\begin{aligned} \cos\beta_x &= \mathbf{n}_s \cdot \mathbf{n}_{sA} \\ \sin\beta_x &= \mathbf{n}_s \cdot \mathbf{b}_{sA} \end{aligned}\right\} , \tag{32}$$

where β_x is the rotation of the cross section relative to a reference plane which is assumed to be at node A, $\mathbf{n}_{sA}$ and $\mathbf{b}_{sA}$ are the unit vectors that define the reference plane at node A. It can be shown using Eq. (32) that

$$\left(\frac{\partial \beta_x}{\partial x}\right)^2 = \left(\mathbf{n}_{sx} \cdot \mathbf{n}_{sA}\right)^2 + \left(\mathbf{n}_{sx} \cdot \mathbf{b}_{sA}\right)^2 , \tag{33}$$

where $\mathbf{n}_{sx} = \dfrac{\partial \mathbf{n}_s}{\partial x}$. The shear angles can be defined as the rotations of the beam cross section about the normal and binormal vectors of Serret-Frenet frame. Considering Fig. 6b, the shear angle β_y can be defined as

$$\beta_y = \sin^{-1}\left(\mathbf{b}_s \cdot \mathbf{t}\right) . \tag{34}$$

Similarly, the shear angle β_z, according to Fig. 6c, is defined as

$$\beta_z = \sin^{-1}\left(-\mathbf{n}_s \cdot \mathbf{t}\right) . \tag{35}$$

Using Eq. (27), the definition of the longitudinal and transverse deformations, the torsion, and the shear angles; the strain energy of the beam can be written as

$$U = \frac{1}{2}\mathbf{e}^T\mathbf{K}\mathbf{e}, \tag{36}$$

where $\mathbf{e}$ is the vector of nodal coordinates, $\mathbf{K}$ is the stiffness matrix. In the current analysis, the beam stiffness matrix is highly nonlinear in the nodal coordinates. Differentiating the strain energy once with respect to the nodal coordinates will lead to the beam elastic forces $\mathbf{Q}$ defined as

$$\mathbf{Q} = -\frac{\partial U}{\partial \mathrm{e}}. \tag{37}$$

In order to ensure that the proposed element meets the convergence requirements, an eigenvalue test has to be performed on the stiffness matrix [7]. The stiffness matrix must have the exact number of rigid body modes (3 for 2-D and 6 for 3-D elements) despite the fact that the stiffness matrix is function of the nodal coordinates. In other words, the stiffness matrix should include exactly 6 zero eigenvalues. If the stiffness matrix has zero eigenvalues more than the number of the rigid body modes, this implies that the element has zero-energy deformation modes which do not have corresponding restoring forces represented in the beam strain energy. Such an element will not converge to a correct solution. Also the stiffness matrix should produce zero strain energy (zero elastic forces) in the case of rigid body motion.

8 NODAL COORDINATES

In the non-incremental absolute nodal coordinate formulation, the displacement field of the finite element is expressed in terms of a set of nodal coordinates that consists of global displacement and slope coordinates. The number of these coordinates depends on the order of the polynomials used. The displacement field presented in Section 2 is assumed to be

linear in y and z and it can assume any order in x. For example, if cubic polynomials in x are used, one needs 24 nodal coordinates for the beam element. This number can be reduced by reducing the order of the polynomials. However, by increasing the order of the polynomials, more deformation modes within one element can be obtained. The use of such a higher order element may lead to a significant reduction in the number of finite elements required to model a structure. Figure 7 shows deformed shapes which can be obtained using one element with polynomials cubic in x. The shape in Fig. 7a is obtained using the following polynomial coefficients:

$$\mathbf{a}_1^T = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -0.27 & 0.018 \end{bmatrix}, \quad \mathbf{b}_1^T = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -0.04 & -0.004 \end{bmatrix},$$

$$\mathbf{c}_1^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -0.11 & 0.004 \end{bmatrix},$$

while the shape presented in Fig. 7b is obtained using the following coefficients:

$$\mathbf{a}_2^T = \begin{bmatrix} 1 & -3 & 0 & 0 & 0 & 0.77 & 0.048 \end{bmatrix}, \quad \mathbf{b}_2^T = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 0 & -0.02 & 0.018 \end{bmatrix},$$

$$\mathbf{c}_2^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -0.13 & 0.002 \end{bmatrix}.$$

Such deformed shapes can not be obtained with one or two elements using linear interpolation. Depending on the order of the polynomial used and the number of nodes selected for the element, one can choose coordinates using the following global variables:

$$\mathbf{r}, \quad \frac{\partial \mathbf{r}}{\partial x}, \quad \frac{\partial \mathbf{r}}{\partial y}, \quad \frac{\partial \mathbf{r}}{\partial z}.$$

These variables represent 12 coordinates at a given point on the center line of the element. One does not need to use all these coordinates at a given node. One may also choose to use another set of coordinates which include a linear combination of the global slopes. For example, the following three variables can be used

$$\vartheta_x = \frac{\partial r_3}{\partial y} - \frac{\partial r_2}{\partial z}, \quad \vartheta_y = \frac{\partial r_1}{\partial z} - \frac{\partial r_3}{\partial x}, \quad \vartheta_z = \frac{\partial r_2}{\partial x} - \frac{\partial r_1}{\partial y}.$$

One can show that in the case of rigid body motion ϑ_x, ϑ_y, and ϑ_z can be related to the known orientation parameters used in the rigid body dynamics. For example, ϑ_x, ϑ_y, and ϑ_z can be expressed in terms of Euler parameters, $\theta_0, \theta_1, \theta_2$, and θ_3, in the case of rigid body motion as follows:

$$\vartheta_x = 4\theta_0\theta_1, \quad \vartheta_y = 4\theta_0\theta_2, \quad \vartheta_z = 4\theta_0\theta_3,$$

and they are related to Rodriguez parameters as follows [18]:

$$\vartheta_x = \frac{4\gamma_1}{1+\gamma^2}, \quad \vartheta_y = \frac{4\gamma_2}{1+\gamma^2}, \quad \vartheta_z = \frac{4\gamma_3}{1+\gamma^2},$$

where γ_1, γ_2, and γ_3 are Rodriguez parameters and $\gamma = \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}$.

A large rotation vector can be defined as $\mathbf{v} \sin\theta$ where $\mathbf{v}$ is a unit vector along the axis of rotation and θ is the angle of rotation. It can be shown that, in the case of rigid body motion, the set $\boldsymbol{\vartheta} = \begin{bmatrix} \vartheta_x & \vartheta_y & \vartheta_z \end{bmatrix}^T$ can be expressed in terms of the rotation vector as

$$\boldsymbol{\vartheta} = 2\mathbf{v}\sin\theta.$$

9 BASIC ASSUMPTIONS

In several finite element formulations for the large rotation and deformation analysis of beams, the equations of motion are developed by assuming that the beam cross section remains rigid [21, 22], while in reality, the beam cross section does not remain rigid when the beam deforms. The use of the assumption of the rigidity of the cross section introduces difficulties when these methods are generalized to the case of plates and shells where the rigidity assumption can not be used and is no longer valid. The non-incremental absolute nodal coordinate formulation relaxes this assumption, and therefore, it can be easily extended to study plate and shell problems.

It was shown in Section 4 that the length of a vector on the cross section of the beam element remains constant under an arbitrary rigid body displacement. In the general case

of arbitrary displacements and deformation, the change in the length of this vector is small and is due to the deformation. It is the objective of this section to demonstrate that this hypothesis is commonly used in the finite element literature. To this end, we use the conventional finite element shape function used for beams and for simplicity we consider the case of two-dimensional beam element. Nonetheless, the main conclusions obtained using the simpler two dimensional beam model apply to the conventional three dimensional beam element used in the incremental procedure.

In the assumed displacement field of the conventional beam element, six nodal coordinates are used for the two-node beam element (3 coordinates/node; two displacement coordinates, and one infinitesimal rotation coordinate). The element shape function of the Euler-Bernoulli beam element is given by

$$\mathbf{S}=\begin{bmatrix} \xi-1 & 0 & 0 & \xi & 0 & 0 \\ 0 & 1-3(\xi)^2+2(\xi)^3 & l\left[\xi-2(\xi)^2+(\xi)^3\right] & 0 & 3(\xi)^2-2(\xi)^3 & l\left[(\xi)^3-(\xi)^2\right] \end{bmatrix}, \tag{38}$$

where $\xi = x/l$, and the vector of nodal coordinates is

$$\mathbf{e}^T=\begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{bmatrix},$$

where

$$\begin{aligned} e_1 &= r_1|_{\xi=0}, \quad e_2 = r_2|_{\xi=0}, \quad e_3 = \left.\frac{\partial r_2}{\partial x}\right|_{\xi=0}, \\ e_4 &= r_1|_{\xi=1}, \quad e_5 = r_2|_{\xi=1}, \quad e_6 = \left.\frac{\partial r_2}{\partial x}\right|_{\xi=1}. \end{aligned}$$

Therefore, the coordinates of an arbitrary point on the center line of the beam element defined in a convected coordinate system is

$$\mathbf{r}=\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}=\mathbf{Se}.$$

Note that this element does not account for the effect of the rotary inertia. In the literature, the effect of the rotary inertia is considered by modifying the assumed displacement field as

follows:

$$\mathbf{r} = \mathbf{Se} + \begin{bmatrix} -\partial r_2/\partial x \\ \partial r_1/\partial x \end{bmatrix} y. \tag{39}$$

Since the shape function of Eq. (38) is independent of y, the vector tangent to the center line is defined as $\partial \mathbf{r}/\partial x$ which can be used to define the normal vector $\mathbf{n} = \begin{bmatrix} -\partial r_2/\partial x & \partial r_1/\partial x \end{bmatrix}$ that appears in the preceding equation. If the element length and the axial deformations are assumed to be small, then

$$\left| \frac{\partial \mathbf{r}}{\partial x} \right| \approx 1. \tag{40}$$

It follows that a vector $\mathbf{r}_s$ on the cross section is defined in the convected coordinates system as

$$\mathbf{r}_s = \begin{bmatrix} -\partial r_2/\partial x \\ \partial r_1/\partial x \end{bmatrix} y = \mathbf{n} y,$$

which implies that $|\mathbf{r}_s|$ =constant if and only if $\mathbf{n}$ is a unit vector. This is not, in general, the case since Eq. (40) is not strictly imposed. To strictly impose the condition of Eq. (40), one must write

$$\mathbf{n} = \frac{1}{\sqrt{(\partial r_1/\partial x)^2 + (\partial r_2/\partial x)^2}} \begin{bmatrix} -\partial r_2/\partial x \\ \partial r_1/\partial x \end{bmatrix},$$

which will produce a shape matrix which is nonlinear in the element nodal coordinates.

10 SUMMARY AND CONCLUSIONS

A non-incremental absolute nodal coordinate formulation for three dimensional beam elements is presented in this paper. The element displacement field is assumed to be linear in y and z coordinates of the cross section of the beam, while it can assume higher order in x. It can be shown that this assumed displacement field can describe exact rigid body motion, and as a consequence, it leads to zero strain energy under an arbitrary rigid body displacement. The kinematics of the element cross section is thoroughly examined and compared

with the Serret-Frenet frame. It is shown that the tangent of the center line of the element does not remain perpendicular to the element cross section, thereby demonstrating that the new element can account for the shear deformation effect as well as the torsion. The shear and torsion angles are defined in terms of the deviation of the element cross section frame from the Serret-Frenet frame. The formulation of the inertia and elastic forces of the finite element using the absolute nodal coordinate formulation was discussed in the paper. It is shown that it is feasible to obtain an element that has a constant mass matrix and at the same time accounts for the rotary inertia, shear, and torsion effects. This property is an important feature of the absolute nodal coordinate formulation since most existing methods used for the nonlinear large rotation analysis, including the incremental methods, large rotation vector formulations, and the floating frame of reference formulation, lead to a highly nonlinear mass matrix when three-dimensional beams are considered.

Global slopes or linear combination of these slopes, in addition to displacement coordinates, can be used as the element nodal coordinates. The number of the element nodal coordinates depends on the order of the interpolating polynomials presented in Section 2. The assumptions used in developing the three dimensional beam element presented in this paper are also discussed and it is demonstrated that these assumptions are consistent with the assumptions that have been used in other methods that employ the conventional three dimensional beam elements. Some other important features of the method proposed in this paper can be summarized as follows:

1. The method, in general, relaxes the assumption of the rigidity of the cross section of the beam element, and therefore, it can be extended to the analysis of plates and shells.

2. The method leads to isoparametric elements that can be easily used in the analysis of curved structures.

3. The method does not require the interpolation of finite rotation coordinates. It is

known that

(a) for a given configuration and a given set of orientation coordinates, different sequences with different values for the orientation coordinates can be used to describe the same configuration.

(b) the relationship between different sets of orientation coordinates is highly nonlinear, and therefore, a linear interpolation of one set of finite rotation coordinates does not imply the same order of approximation for another set. This raises some questions with regard to the physics used in the interpolation of the finite rotation coordinates. On the other hand, in the absolute nodal coordinate formulation, only the shape of the element is interpolated.

4. The proposed method ensures the continuity of the slopes and the rotation of the cross section. As a consequence, the configuration shown in Fig. 8 which can result from the interpolation of rotation coordinates only, is avoided. This configuration is the result of imposing continuity conditions on the finite rotations, while no such conditions are imposed on the global slopes.

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REFERENCES

[1] Argyris, J. H., et. al., 'Finite element method-The natural approach,' *Computer methods in Applied Mechanics and Engineering* **17/18**, 1979, 1-106.

[2] Ashley, H., 'Observation of the dynamic behavior of flexible bodies in orbit,' *AIAA J.* **5**, No. 3, 1967, 460-469.

[3] Bauchau, O. A. and Kang, N. K., 'A multibody formulation for helicopter structural dynamic analysis,' *Journal of the American Helicopter Society*, April 1993, 3-14.

[4] Berzeri, M., and Shabana A. A., 'A continuum mechanics approach for formulating the elastic forces in the absolute nodal coordinate formulation,' *Technical Report No. MBS99-3-UIC*, University of Illinois at Chicago, 1999.

[5] Burke, W. L., *Applied Differential Geometry*, Cambridge University Press, Cambridge, 1985.

[6] Canavin, J. R., and Likins, P. W., 'Floating reference frames for flexible spacecraft,' *Journal of Spacecraft* **14**, No. 12, 1977, 724-732.

[7] Cook, R. D., Malkus, D. S., and Plesha, M. E., *Concepts and Applications of Finite Element Analysis*, 3rd edition, John Wiley & Sons, 1989.

[8] Cowper, G. R., ' The shear coefficient in Timoshenko beam theory' *Journal of Applied Mechanics* **33**, 335-340.

[9] de Veubeke, B. J., 'The dynamics of flexible bodies,' *International. Journal of Engineering Sciences* **14**, 1976, 895-913.

[10] Dym, C. L., and Shames, I. H., *Solid Mechanics, A Variational Approach*, McGraw-Hill, 1973.

[11] Escalona, J. L., Hussien, A. H., and Shabana, A. A., 'Application of the absolute nodal co-ordinate formulation to multibody system dynamics,' *Journal of Sound and Vibration* **214**, No. 5, 1998, 833-851.

[12] Greenberg, M. D., *Advanced Engineering Mathematics*, Prentice Hall, 1988.

[13] Kane, T. R., and Levinson, D. A., 'Simulation of large motions of nonuniform beams in orbit: Part II-The unstriated beam,' *The Journal of the Astronautical Sciences* **29**, No. 3, 213-244.

[14] Przemieniecki, J. S., *Theory of Matrix Structural Analysis*, 2nd edition, Dover, 1985.

[15] Rankin, C. C., and Brogan, F. A., 'An element independent corotational procedure for the treatment of large rotations,' *Journal of Pressure Vessel Technology* **108**, 1986, 165-174.

[16] Shabana, A. A.,'An absolute nodal coordinate formulation for the large rotation and deformation analysis of flexible bodies,' *Technical Report No. MBS96-1-UIC*, University of Illinois at Chicago, 1996.

[17] Shabana, A. A., 'Flexible multibody dynamics: Review of past and recent developments,' *Multibody System Dynamics* **1**, 1997, 189-222.

[18] Shabana, A. A., *Dynamics of Multibody Systems, 2nd edition,* Cambridge University Press, Cambridge, 1998.

[19] Shabana, A. A., 'Computer implementation of the absolute nodal coordinate formulation for flexible multibody dynamics,' *Nonlinear Dynamics* **16**, 1998, 293-306.

[20] Shabana, A. A., Hussien, H. A., and Escalona, J. L., 'Application of the absolute nodal coordinate formulation to large rotation and large deformation problems,' *ASME Journal of Mechanical Design* **120**, 1998, 188-195.

[21] Simo, J. C., 'A finite strain beam formulation. The three dimensional dynamics problem. Part I,' *Journal of Computer methods in Applied Mechanics and Engineering* **49**, 1985, 55-70.

[22] Simo, J. C. and Vu-Quoc, L., 'On the dynamics of flexible beams under large overall motion-the plane case: Part I,' *ASME Journal of Applied Mechanics* **53**, 1986, 849-854

[23] Takahashi, Y. and Shimizu, N., 'Study on elastic forces of the absolute nodal coordinate formulation for deformable beams,' presented at the *Second Symposium on Multibody Dynamics and Vibration, ASME Conferences*, Las Vegas, Sept. 1999, Paper # DETC99/VIB-8203.

[24] Yakoub, R. Y. and Shabana, A. A., 'Use of Cholesky coordinates and the absolute nodal coordinate formulation in the computer simulation of flexible multibody systems,' *Journal of Nonlinear Dynamics* **20**, 1999, 267-282.

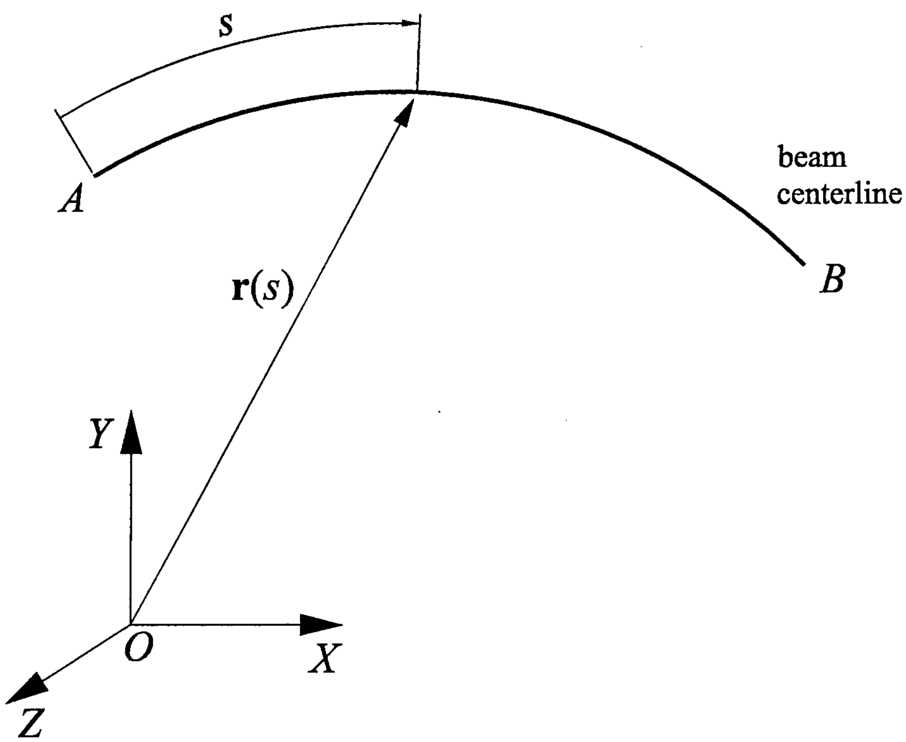


Fig. 1. Beam rpresented by its center line.

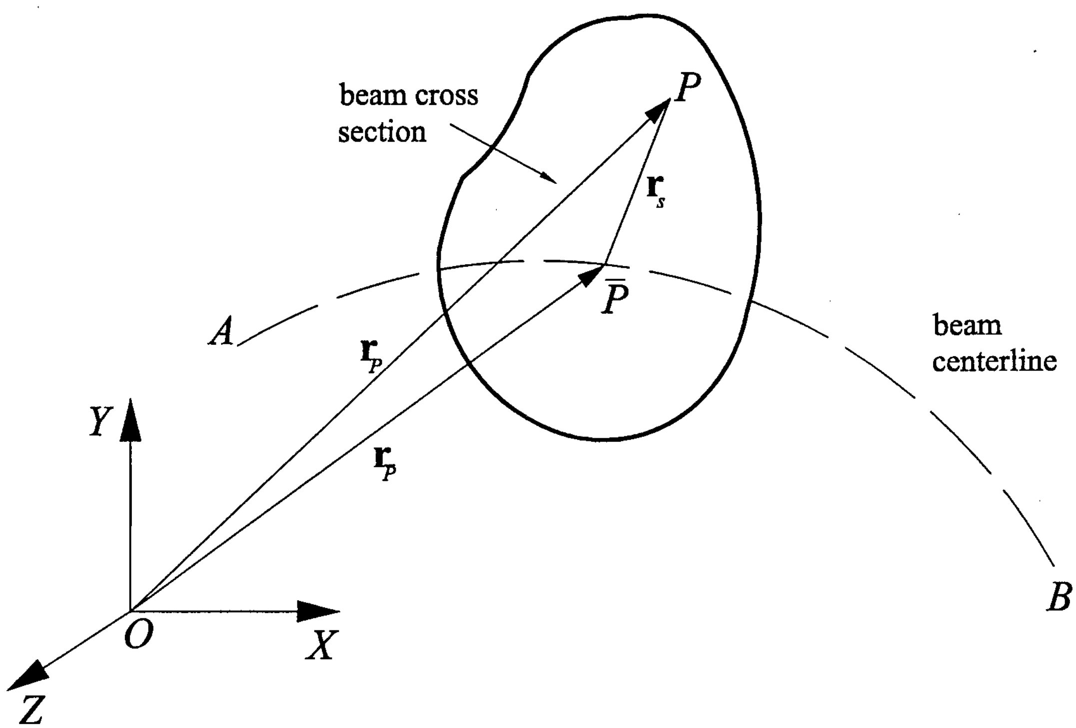


Fig. 2. Beam cross section.

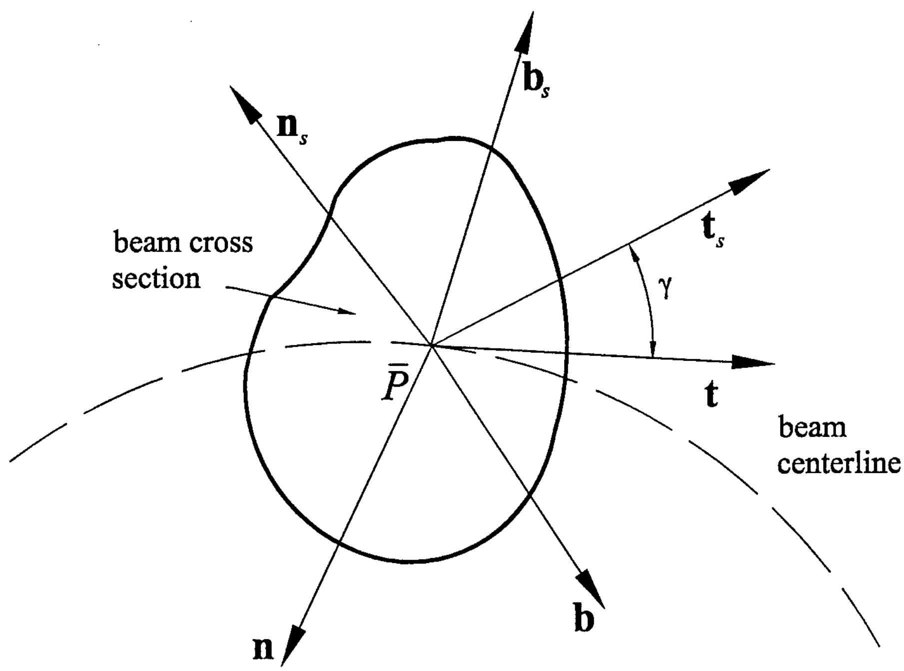


Fig. 3. Coordinate systems.

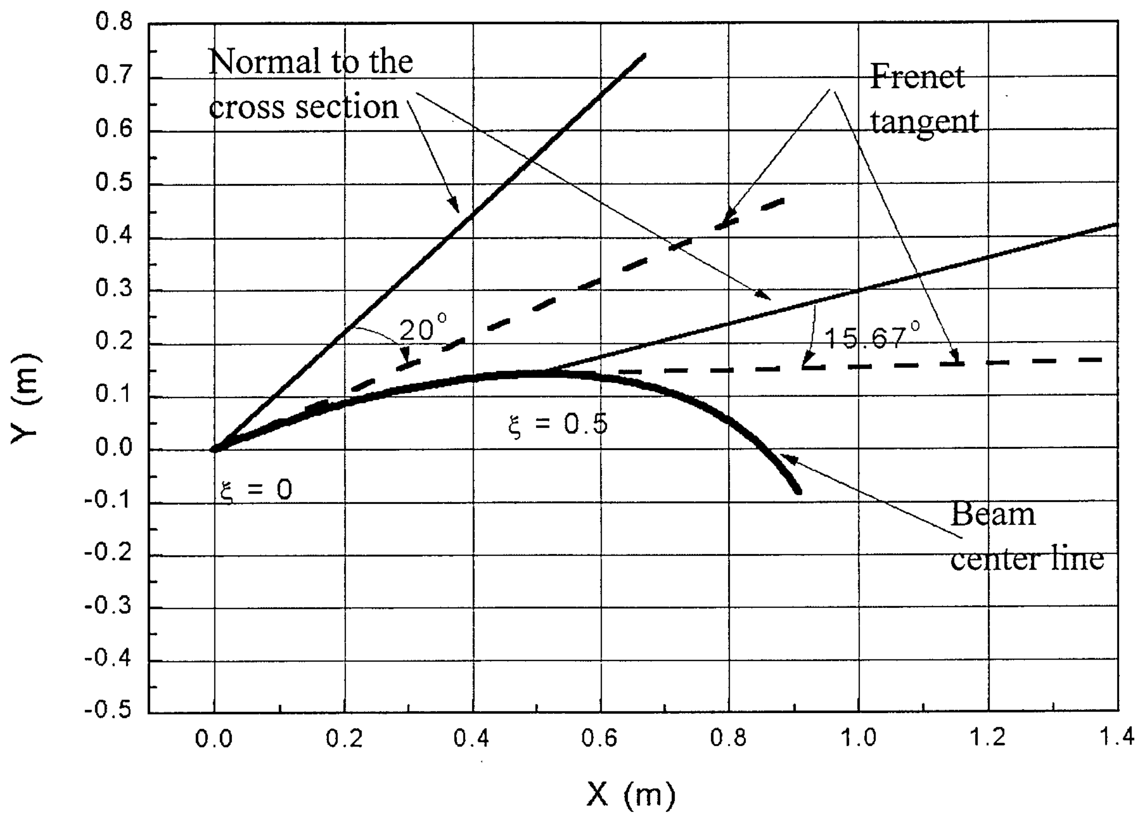


Fig. 4. Rotation of the beam cross section.

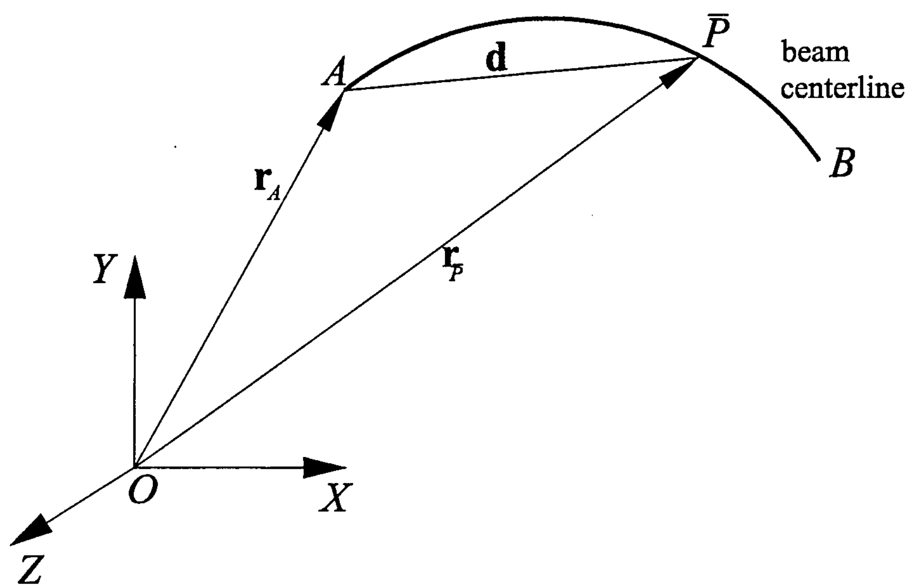
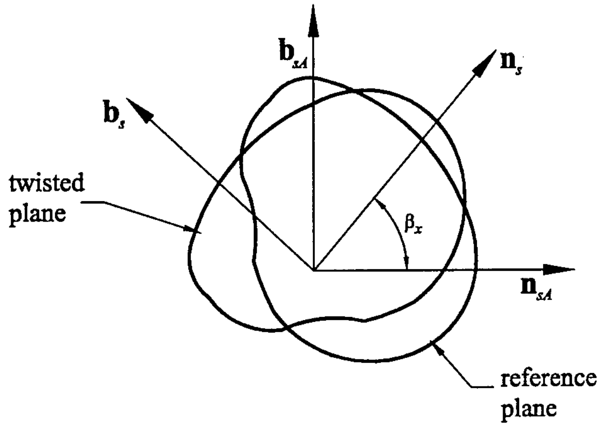
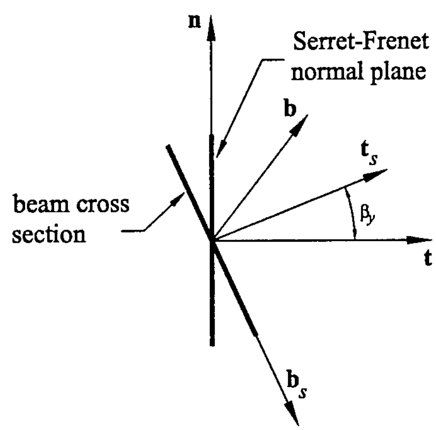


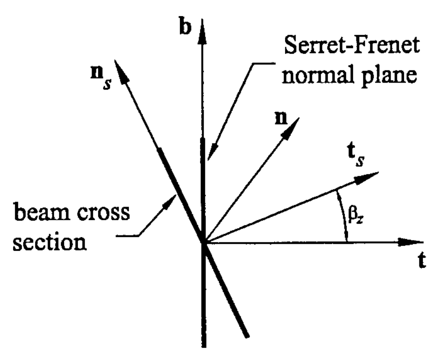
Fig. 5. Beam deformation.



(a) Rotation of beam cross section about its normal.

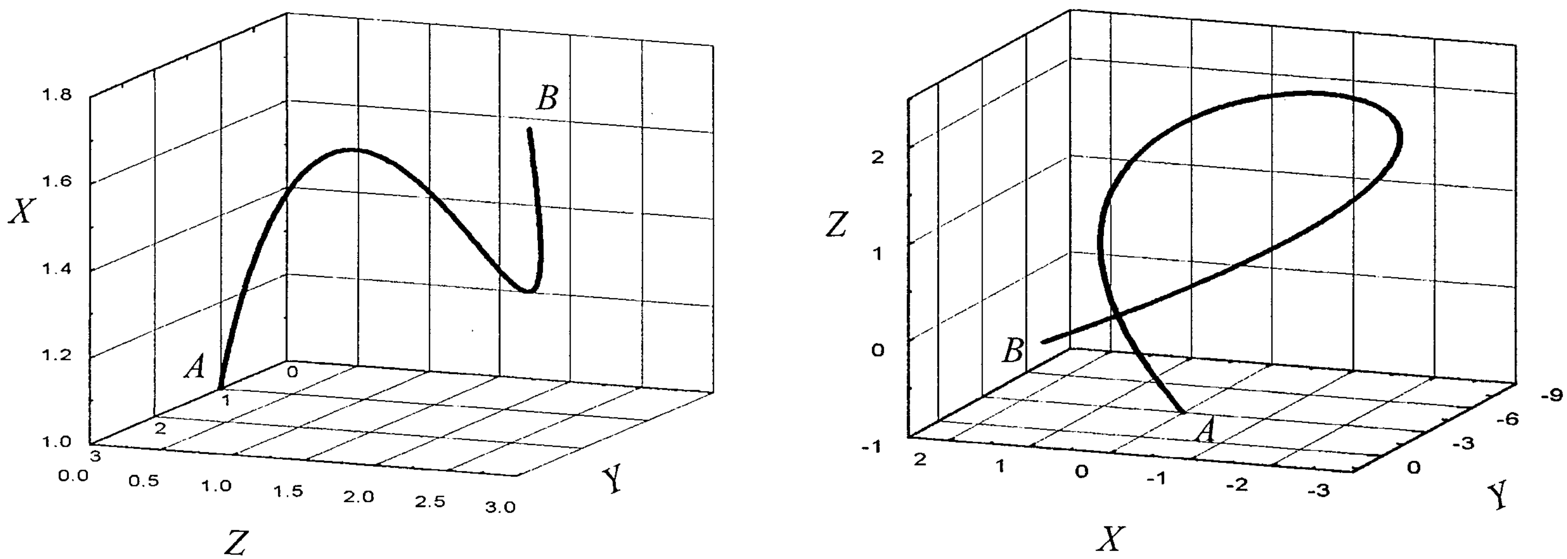


(b) Rotation of beam cross section due to shear β_y



(c) Rotation of beam cross section due to shear β_z

Fig. 6. Definition of torsion and shear.



(a)

(b)

Fig. 7. Different beam shapes using cubic polynomials.

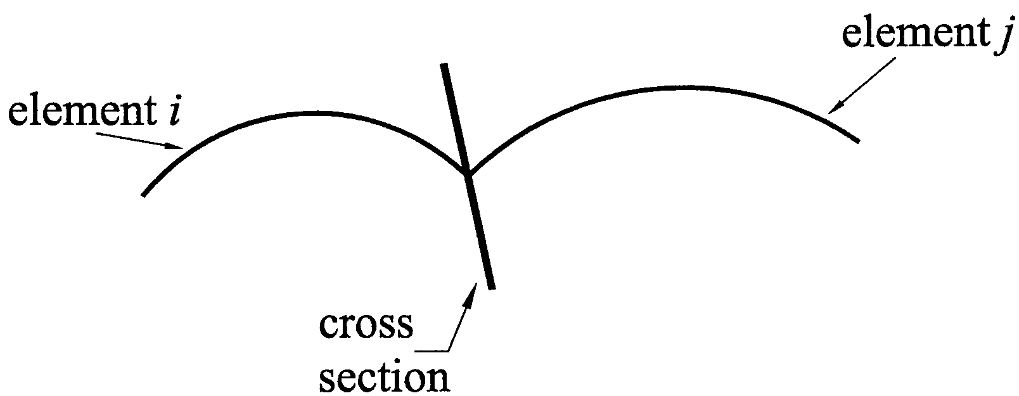


Fig. 8. Discontinuity of the slopes.